ON THE STABILITY OF PLANE-PARALLEL FLOW OF A VISCOUS FLUID OVER AN INCLINED BOTTOM

(OB USTOICHIVOSTI PLOSKOPARALLEL'NOGO TECHENIIA VIAZKOI ZHIDKOSTI NAD NAKLONNYM DNOM)

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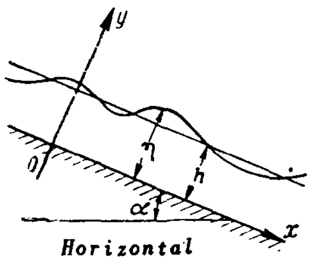
Consider a fluid bounded above by a free surface and below by a rigid plane inclined at an angle α to the horizontal (see figure).

We shall assume that the flow proceeds in the direction of the x-axis, whilst the y-axis is along the upward normal. After rendering the variables dimensionless by reference to the mean depth of the stream h and the acceleration due to gravity g, we have the following equations describing the motion of the viscous fluid [1]:



$$\frac{\partial^2\psi}{\partial t\partial y}+\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=-p_x+\sin\alpha+\frac{1}{H}\frac{\partial}{\partial y}\Delta\psi \tag{1}$$

$$\frac{\partial^2\psi}{\partial t\partial x}+\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x^2}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial y}=p_y+\cos\alpha+\frac{1}{H}\frac{\partial}{\partial x}\Delta\psi$$

$$u=\frac{\partial\psi}{\partial y},\qquad v=-\frac{\partial\psi}{\partial x},\qquad H=\frac{\sqrt{gh}h}{\nu}$$

Here ψ is the stream function, u and v are the components of velocity parallel to the x- and y-axes respectively, p is the pressure, and H is the "viscous depth".

Eliminating p from Equations (1) we obtain the following equation for the stream function:

$$\frac{\partial\Delta\psi}{\partial t}+\frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y}=\frac{1}{H}\Delta\Delta\psi \tag{2}$$

At the bottom we have two kinematic boundary conditions

$$\psi=\text{const},\qquad \frac{\partial\psi}{\partial y}=0\quad\text{when}\quad y=0 \tag{3}$$

At the surface $y = \eta$, we have one kinematic

$$\eta_t + \frac{\partial\psi}{\partial y}\eta_x = -\frac{\partial\psi}{\partial x} \tag{4}$$

and two dynamic conditions

$$-p + \frac{2}{H}\frac{\partial^2\psi}{\partial x\partial y}\frac{1-\eta_x^2}{1+\eta_x^2} + \frac{2}{H}\left(-\frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial x^2}\right)\frac{\eta_x}{1+\eta_x^2} = \text{const} \tag{5}$$

$$\left(-\frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial x^2}\right)\frac{1-\eta_x^2}{1+\eta_x^2} + 4\frac{\partial^2\psi}{\partial x\partial y}\frac{\eta_x}{1+\eta_x^2} = 0 \tag{6}$$

It is well known that an exact solution of Equations (1) is the flow parallel to the x-axis

$$\psi_0 = \frac{H}{2}\sin\alpha\left(y^2 - \frac{1}{3}y^3\right), \qquad p_0 = \cos\alpha\,(y-1)$$

with the discharge (q in dimensional variables)

$$Q = \psi_0\,|_{y=0} = \frac{H\sin\alpha}{3}, \quad \text{or} \quad q = \frac{gh^3\sin\alpha}{3\nu}$$

It is not difficult, however, to show that such a flow is unstable for a certain relation between H and α. Let us set

$$\psi = \psi_0 + \psi_1, \qquad p = p_0 + p_1, \qquad \eta = 1 + \eta_1.$$

where ψ_1, p_1 and η_1 are certain small perturbations. We shall consider perturbations of long wave type:

$$\psi_1 = \varphi(y)\,e^{i\varepsilon(x-ct)}, \qquad \eta_1 = ne^{i\varepsilon(x-ct)} \tag{7}$$

By virtue of the assumptions which have been made concerning the character of the perturbations, ϵ is a small quantity. Such an assumption is physically justifiable, since in "viscous" media oscillations with high frequency (short wave-length) are quickly damped out.

Let us substitute (7) into Equation (1) and into the boundary conditions (3)-(6) (and let condition (5) be first differentiated along the free surface). Discarding terms of the second order of smallness, we have for $\phi(y)$ the ordinary differential equation

$$\varphi^{IV} + i\varepsilon H\left(c - \frac{d\psi_0}{dy}\right)\varphi'' + i\varepsilon H\frac{d^3\psi_0}{dy^3}\,\varphi = 0 \tag{8}$$

with the boundary conditions

$$\begin{gathered}\varphi(0)=0,\ \varphi'(0)=0, \qquad \varphi'''(1) + i\varepsilon(c-{}^1/_2H\sin\alpha)\,H\varphi'(1) = ni\varepsilon H\cos\alpha \\ \varphi(1) = n\,(c - {}^1/_2\,H\sin\alpha), \qquad \varphi''(1) = nH\sin\alpha\end{gathered} \tag{9}$$

For the sake of brevity in the equations we do not write down terms $O(\epsilon^2)$, since the solution of Equation (8) will be sought in the form of a power series in ϵ in which we shall restrict ourselves to the first power. This is legitimate by virtue of the fact that the equation does not contain ϵ in the highest derivative and the boundary conditions do not degenerate when $\epsilon \to 0$.

The general solution of Equation (8) can be written as

$$\varphi = C_1\varphi_1 + C_2\varphi_2 + C_3\varphi_3 + C_4\varphi_4 \tag{10}$$

where ϕ_i are four linearly independent particular solutions of Equation (8); for these functions, let us take

$$\begin{aligned} \varphi_1 &= 1 + O(\varepsilon), \quad \varphi_3 = y^2 - \frac{i\varepsilon Hcy^4}{12} + \frac{i\varepsilon H^2 \sin \alpha y^5}{60} \\ \varphi_2 &= y + O(\varepsilon), \quad \varphi_4 = y^3 - \frac{i\varepsilon Hcy^5}{20} + \frac{i\varepsilon H^2 \sin \alpha y^6}{60} - \frac{i\varepsilon H^2 \sin \alpha}{420}\, y^7 \end{aligned} \tag{11}$$

By virtue of the first two of conditions (9), we have $C_1 = C_2 = 0$. After substituting (10) and (11) in the remaining boundary conditions we obtain a system of three linear equations relating the three unknowns C_3, C_4, and n. For the system to be soluble it is necessary and sufficient that the determinant of the equations

$$\begin{vmatrix} 1 - \frac{i\varepsilon Hc}{12} + \frac{i\varepsilon H^2 \sin\alpha}{60} & 1 - \frac{i\varepsilon Hc}{20} + \frac{i\varepsilon H^2 \sin\alpha}{70} & c - \frac{H \sin\alpha}{2} \\ 0 & 6 & i\varepsilon H \cos\alpha \\ 2 - i\varepsilon Hc + \frac{i\varepsilon H^2 \sin\alpha}{3} & 6 - i\varepsilon Hc + \frac{2i\varepsilon H^2 \sin\alpha}{5} & H \sin\alpha \end{vmatrix} = 0$$

Substituting $c = F + i\lambda$ and separating the real and imaginary parts, we obtain (to accuracy ϵ)

$$F = H \sin\alpha, \quad \lambda = \tfrac{2}{15}\varepsilon H^3 \sin^2\alpha - \tfrac{1}{3}\varepsilon H \cos\alpha \qquad \left(F = \frac{U}{\sqrt{gh}}\right)$$

Here F is the Froude number. The first condition relates the velocity of propagation of the wave to the depth of the fluid

$$U = \frac{gh^3 \sin\alpha}{\nu} = 3u_m = 2u_{\max}$$

Here u_m and $u_{\max}$ are the mean and maximum velocities of the parallel flow. The parallel flow is unstable if $\lambda > 0$, i.e.

$$\tfrac{2}{5} H^2 \sin^2\alpha \geqslant \cos\alpha$$

With allowance for the capillary effect the condition for instability takes the form

$$-\cos\alpha + \varepsilon^2\sigma_1 \pm \tfrac{2}{5} H^2 \sin^2\alpha \geqslant 0$$

where σ_1 is the dimensionless surface-tension coefficient and $2\pi/\epsilon$ is the dimensionless wave-length.

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